

The Wave Function and i

*A personal exploration
of the mechanics of quantum physics*

Roy Freeman

September 6, 2010

Contents

1	What this exploration is about (and what not)	2
1.1	About: the wave equation	2
1.2	About: i and non-cummutivity	2
1.3	About: the roots and limitations of our knowledge of nature	3
1.4	What this is <i>not</i> about	4
1.5	My approach	4
2	QM: The basics	5
2.1	The Dirac-Feynman formulation of QM	5
2.2	Beginning at the beginning	6
2.3	Mathematics: numbers (complex numbers), matrices, and vectors	8
2.3.1	Vectors	9
2.3.2	Numbers and complex numbers	11
2.3.3	Complex numbers	12
2.3.4	Some other special numbers in QM	13
2.4	Feynman's double-slit experiment	14
2.5	Discussion of Feynman's example	17
2.6	What is the wave equation about?	19
2.7	Equations and differential equations	19
2.7.1	Scalar equations	19
2.7.2	Quadratic equations	19
3	Appendix	21

1 What this exploration is about (and what not)

1.1 About: the wave equation

1.2 About: i and non-cummutitivity

What is really different about quantum mechanics from previous physics? The table below compares the fundamental equations of classical with quantum mechanics:

Classical	Quantum
$[pq - qp] = 0$	$[pq - qp] = i\hbar$
$\partial\Psi = H$	$\partial\Psi = iH$

Table 1.1: Comparison of the “equations of motion” classical and quantum mechanics

We will look at these equations in more detail below. It suffices now to note that the right hand side of the basic equation of quantum mechanics:

$$[pq - qp] = i\hbar \quad (1.1)$$

is not zero as in its classical counterpart, but $i\hbar$; and that the right hand side of the wave equation of quantum mechanics

$$\partial\Psi = iH \quad (1.2)$$

is iH and not simply H .

These are two, perhaps innocent-looking, radical differences:

1. the mathematics of QM as evidenced by the right hand side of eq 01, are non-cummutative (otherwise a zero would be there as in the classical counterpart, and
2. the imaginary unit i (and therefore complex algebra) is explicitly included in both quantum mechanical equations.

It need not bother you if you have never heard of the wave function, non-cummutivity or i , this exploration will take you step-by-step to meet these entities; it is about where these two innovations come from, why they *have* to be in the equations, and what they have to tell us about nature and our knowledge of nature. To get to the very center, we have to remain simple and steadfast. You have the fish (diamonds) in your hands (well, two fish) now do not let them go!

If one follows the quantum mechanical equations rigorously, they lead to the conclusion that not only the observer, but in fact all parts of the system are “entangled”. This disturbed Einstein and he protested¹ but his arguments were proven with experiments to

¹int the paper by Einstein, Podolsky, and Rosenberg

confirm this. But where does this “entanglement” come in? It comes from the differences in the fundamental equations noted in the table above, from i , the wave equation, and from non-commutivity, our should I say: it is how Nature really is, whether we understand it or not? One thing I wonder about is where is the feminine in QM? As far as I know (I would be happy if someone were to correct me!) no woman has so far contributed significantly to the development of the *formalism* of QM theory. Perhaps women know that the world is entangled, and need no big mathematical theory to tell them what they already know and live every day. As a firm appreciator of the outstanding – and often superior thinking in terms of networked – qualities of the other sex, (even if she thinks differently than I) I distrust a theory that is too unisexual.

One the other hands, perhaps the indeterminacy (of the state function) that QM lays on theories of science about nature is a sort of inclusion of the feminine into man’s way of thinking. Indeterminacy was one of the major innovative relative to classical mechanics and has far-reaching consequences for science. The male world of rational science has to move in the direction of the female in order to describe nature correctly! Perhaps in writing this I am seeking the woman in QM.

1.3 About: the roots and limitations of our knowledge of nature

Some history: At the Solvay Conference of 1927,² Niels Bohr, Werner Heisenberg, Wolfgang Pauli, Paul Dirac and Max Born³ came together to clarify just what the “quantum mechanics” that they had helped to develop was really about. After this meeting Dirac summarized the result: “The wave function represents our knowledge of the system.” (Hendry (1984). The mathematical theory was about our knowledge of physical reality and not about physical reality itself, apart from the observer. To make this clear, here are two quotes by Heisenberg ?

The conception of objective reality of the elementary particles has thus evaporated not in to the cloud of some obscure new reality concept but into the transparent clarity of a mathematics that represents no longer the behavior of particles but rather our knowledge of this behavior.

and

“... the act of registration of the result in the mind of the observer. The discontinuous change in the probability function [the wave function] ... takes place with the act of registrations, because it is the discontinuous change in our knowledge in the instant of registration that has its image in the discontinuous change in the probability function.”

² The International Solvay Institutes for Physics and Chemistry, located in Brussels, were founded by the Belgian industrialist Ernest Solvay in 1912. Following the initial success of 1911, the Solvay Conferences have been devoted to outstanding preeminent open problems in both physics and chemistry. Perhaps the most famous conference was the October 1927 Fifth Solvay International Conference on Electrons and Photons, where the world’s most notable physicists met to discuss the newly formulated quantum theory.

³Evidently Erwin Schrödinger, although the “creator” of the wave equation, was not a part of this group, he did not like the “jumps” that the initial equations introduced.

On the other hand, calculations of atomic phenomena using the quantum mechanic formalism⁴ has yielded the most accurate description of nature in the small that we know. Now how can a theory about our (*subjective*) *knowledge* simultaneously yield such accurate answers about *objective nature*? Evidently, it is in the mathematics, this being about as removed from our subjectivity as we can imagine. (Perhaps it is somewhere else. As a natural scientist, I must admit that I do not know the answer, but like trying to find it.)

1.4 What this is *not* about

This is not a recipe for calculating atomic orbitals or chemical bond energies. There are excellent books on this topic.⁵ Meanwhile there are even programs on the Internet to do this (in a simplified way).

It is also not about popular ideas of quantum coherence, quantum non-locality, or Bell's inequality. However, it *is* about where these come from in the formalism of quantum mechanics.

1.5 My approach

My premise is that something in the mathematical formalism of QM contains gems. Therefore this is going to be an introduction to the mathematical formalism following that which P.A.M. Dirac set out in his book *Principles of Quantum Mechanics* Dirac [1958]. Dirac's formulation is elegant and very concise. I sense that it contains the whole story – somewhere. However, Dirac is almost *too* concise for me, here I fill out his description with my own explanations. Actually, I think it is my version of *Quantum Mechanics for Dummies*, the dummy being me of course!

I assume that you the reader has very little experience with mathematics, but that you are interested and willing to follow me wherever I go. Some of the mathematical apparatus do not get me closer to my goal and I take the liberty of bringing these in without much explanation. There are excellent books on the subject. Other mathematical points seem to flicker or glow slightly as if indicating that something deeper is there. I will take more time with these points (e.g. complex numbers). In keeping with this, I relegate details of concepts that are necessary to QM (but not to what I think are deeper levels) to the [Appendix](#). If you read something in blue, then it means this phrase or topic is covered in more detail in the Appendix. To get there, just click on the blue phrase, like: Here is an example: [Dirac brackets](#)

⁴*Quantum electrodynamics*, a further development of quantum mechanics pioneered by Richard Feynman and Julian Schwinger.

⁵Hans Primas at the ETH Zürich has published an excellent concise student version.

2 QM: The basics

Quantum Mechanics (QM) is a formalism, that is, a collection of mathematical formulas and rules with their physical interpretation, for calculating what physicists say that we can know about nature. The word *quantum* is Latin for “how great” or “how much.” In physics since 1900, quantum refers to a discrete value assigned to certain physical quantities, such as the energy of an atom at rest. On the other hand, quantum mechanics since Schrödinger’s illumination in 1928 is now associated with a wave function. Normal waves go smoothly up and down, they do not jump around. So here we have a real paradox. For the moment, suffice to say that the waves of quantum mechanics have nothing to do with waves in the ocean, they are mathematical waves! But they really go up and down! In order to understand this – and it is fundamental for my purpose in this exploration – I will tarry on this point hopefully until everybody at least understands the difference.

Exact experiments demonstrate that the movement of a small particle – like an electron bound in a system of a molecule – looks more like a wave on the surface of a pond than the scratch of a dot on some plate (Martin Gutzwiller [2007]).⁶ Actually QM teaches us that we have to formulate our questions and answers more precisely: when we make an experiment to find out *where* a light is, then the experimental results can be best described (note that I do not use the word “explained”) when we assume that light behaves (at least when we ask where it is) as if it were a wave. If we ask “what time did you pass through my measurement apparatus? Then light answers as if it were a discrete particle (a discrete packet of energy).

QM is an interesting theory. It is a mathematical formalism for calculations that match experimental results better than any other physical theory. It is at the same time a statement about how much we can *know* about nature. Thus it appears to be about nature and/or about how we look at nature, that is, about human consciousness. Evidently – and this is what is so significant – QM says that we can not separate these two things. This is what makes it so interesting (to me). In order to explore the inner working of such a far-reaching theory, we have to understand the mathematics. There is now way of getting around the details, so let’s go!

2.1 The Dirac-Feynman formulation of QM

The Dirac formulation (Dirac [1958]) expresses the mathematics of QM very concisely. This conciseness makes it look simple and is certainly elegant and you could even say, beautiful (we can discuss another time what is beautiful in science), but is hard to get a practical hand on if you have not already studied QM. This I would like to remedy, since Dirac’s formulation brings out the crucial points of QM very clearly. Richard Feynman in his published lectures Feynman et al. [1965a] uses Dirac’s notation in a wonderfully simple and easy-going way. I will follow Feynman’s path to help understand the practical side of Dirac’s formulation. After that, we will know enough of QM to at least *ask* what it is all about. To get a bit closer to the core, there is no other way than to turn to nature, that is experiments. It was actually experimental results that forced the birth and development

⁶Gutzwiller also writes: “These waves obey linear partial differential equations, whose solutions have smooth shapes, and are quite pleasant to behold.” Although I appreciate beautiful shapes, I am here more interested in the inside workings of these shapes.

of QM. This is a fundamentally important fact. Therefore, we will apply QM to the structure of the simplest atom, hydrogen. For this we turn to the wave formulation, a bit easier to practically apply, but smudges over some of the underlying points that make QM what it is.

2.2 Beginning at the beginning

Tools to describe what we (can) know about nature

We experience nature around us and within us directly every moment of our lives. But humans evidently have a special quality, they not only want to experience things, they also want to know about what they experience. As far as I am aware, no other species living today has this quality. But how do we know something? (Do we know something we do not experience consciously, (even if our sense receptors have registered the incoming signals?) A certain pattern of synapses firing at different regions of the brain at the same time connect an unknown pattern with a known pattern. Evidently we have to have a known pattern before we can explore the edges or what could be new. We have to organize some of these synapse networks and I will do this by gathering patterns of things we know already or think we know). And to make sure we all mean more-or-less the same image when, I will try to define some of these patterns. Naturally I will use the image that science has already decided to use, so that we can speak with one another, read other literature on QM.

Curiously, QM does not concern itself with laws, but rather formalizes ways to connect certain conceptual entities related to how and what questions we can ask nature. *Ask* in the sense of *make an experiment* to get a quantitative (numerical) value related to some physical (conceptual) entity.

QM describes nature in terms of systems, states, and operators. (I will define these below.) The nature that QM describes is in terms of (physical entities) particles, waves, and forces. QM uses mathematical entities to describe the relationships between these two groups. These mathematical entities are: numbers, vectors, and matrices. QM is

Nature	Mathematics	QM formalism
systems	matrices	systems
states	vectors	waves/particles
forces (interactions)	numbers	operators

Table 2.1: The concepts and tools that QM uses to describe nature

mostly interested in the changes that a physical object experiences. To describe changes, QM considers *where* something is (in our physical three-dimensional space) and *when* something happened (in time). To be a bit more precise, QM describes *changes* in position and *changes* in time (how fast or slow something changes). Other features are:

color (energy)⁷, weight (mass), wavelength (frequency), charge, and spin.

Physics: The state of a system

Take a molecule, say water, H₂O, a system composed of two hydrogen atoms (H₂) and one oxygen atom (O). The system of a water molecule is actually no longer two hydrogens and one oxygen; the electron of each hydrogen is being shared with the electrons of oxygen in one cloud. The hydrogen atom is now a single proton as part of a water molecule (an oxygen atom and another hydrogen proton. (QM tells us that we cannot distinguish this “ex-hydrogen 1 proton” from the “ex-hydrogen 2 proton” although we can somewhat localize its relative position.) The electrons, neutrons, and protons are in turn composed of quarks, their state is still another microsystem exchanging energy through photons. All these particles have mass (weight, even if very small); they are moving, so they have inertia; they are interacting with each other depending on different kinds and strengths of forces. Each movement is called a *state* of that system. If we are looking at only an electron, the motion of that electron under the different forces is the state of that electron. If we consider the whole water molecule with all the jiggles and goings-on inside, the system would be the whole molecule.

Classical physics would say that understanding the system is equivalent to specifying a numerical value of all the coordinates and velocities of the various component parts of the system at some particular instant in time. Things are not quite so simple in QM.

To completely describe what is going on, we need concepts for what is moving (particles, waves, for instance) and what moves them (forces).

Physics: particles, waves, and forces

Physics describe what *is*, with things like “how much does it weigh?” or “What is its charge?” In other words, the properties that something has. At the outset of QM (the early 1900’s), physicists thought they knew all of the properties that a thing must have to be characterized. This turned out to be false. In the course of the development of QM, several intrinsic properties of the kinds of things in this universe had to be discovered. *Spin* is property unknown before QM that every particle in the universe has, even if its spin = 0! (I will go into in more detail about spin below.) Not only a single particle, but also an aggregate of particles, like an atom, or a molecule has a total “spin”.

Here we meet a subtle conceptual change that QM introduced. Note that I wrote “all particles have spin.” Some particles (photons for instance) have spin = 0. But in QM we cannot say “a photon does not have spin”; it does have spin, its spin is, however, zero. This precision of what “nothing”, “hole”, “absence” means, is a fundamental broadening of our understanding of nature. But this is not actually new to human experience. Something missing can have an effect. (Think of emotionally, not getting love; that can have life-long effects.) In physics if a particle has spin = 0 this does not mean the particle has no spin, it means exactly what it says: “the particle has spin of zero”. Funny, right? Well, interestingly enough, we humans experience something similar in daily life. Having

QM began with the fact that experimental results could not be explained.

In fact, it is the broadening of concepts – an intellectual change – that is one of the great insights (perhaps “gifts”) that QM offers.

⁷Light is energy. Blue light transfers greater energy than red light; blue light has a wavelength of around 475 nm (= 10⁹) m, red light has a larger wavelength (around 600-700 nm) and thus the frequency (wavelengths per unit time) is larger for blue light. Numerically, photon energy (E) = Planck’s constant (h) × speed of light (c) / wavelength (λ). With $c = 3 \times 10^8$ meters/second and $h = 6.626 \times 10^{-34}$ joules x seconds then $E(\text{blue}) = 4.2 \times 10^{-19} \text{ J} = 2.6 \text{ eV}$ and $E(\text{red}) = 3.1 \times 10^{-19} \text{ J} = 1.9 \text{ eV}$ (eV = electron volts).

“nothing” is still having something, as Janis Joplin sings in *Me and Bobby McGee*⁸ I might even venture that zero in physics does not necessarily have, at least superficial, the same meaning as 0 in mathematics. Of course at basis, it must have the same meaning, so we must either expand physics, or expand math, or expand our understanding of both!

No one has ever seen a particle “spinning”, no one really knows what this property of spin physically is. However, without it, our description and understanding of the world is incomplete. The spin of a packet of energy (a photon, for instance) has very practical consequences. Laser light would not be possible if photons did not have spin = zero.

forces →
operators in
QM

A particle can also be better described as a wave, depending on how we look at it. Also what moves particles or wave is a basic physical concept. As of today, there are four forces in the universe: electromagnetic, gravitation, strong, and weak forces. QM does not say any new things about the forces, it calls them *operators* and formulates how we related these to describing the states of a system. Forces, properties, velocities, inertia, etc are the dynamic variables of a system.

Dirac states (Dirac [1958] p. 15):

“QM requires that the states of a dynamical system and the dynamical variables be interconnect in quite strange ways that are unintelligible from the classical standpoint. The states and dynamic variables have to be represented by mathematical quantities of different natures from those ordinarily used in physics. A new scheme requires axioms and rules to deal with the (new) mathematical quantities are specified and addition certain laws connecting physical facts with mathematical formulation or defined so that from any physical condition, equations between the mathematical quantities may be inferred and vice versa.”

2.3 Mathematics: numbers (complex numbers), matrices, and vectors

The relationship between physics and mathematics is somewhat like a relationship between a man and woman, different but madly attracted to each other; impossible to reconcile, but cannot live apart. Mathematics in QM turns out to be not only a very efficient method of notekeeping, but intrinsic to the understanding, so you really cannot (at least in my opinion) divide the math from the physics.

To work, physics needs numbers in their qualitative sense to give a meaning comparable to magnitude to a parameter (how big, how long, how fast). The qualitative (ordering, cardinal) sense of numbers also plays a role in QM. This is more subtle and I will get more attention later.

⁸“And nothing aint worth nothin’, but its free.” Lyrics by Fred Foster and Kris Kristofferson.

2.3.1 Vectors

Vectors are a mathematical way of expressing something that has quantity and quality. The quality here is *direction*, probably not what most readers think of as quality, but here think of an extra dimension: a number by itself does not have a direction. Vectors add this “quality” by a formalism, remember what that is? Now, interestingly, this is how our brains work. Our brain “works” with vectors (see appendix ?? and calculates with matrices (coming right below)).⁹ Physics also needs a mathematical representation for states, operators, and things that are operated on in states, called. The formalism best suited to Dirac’s representation is to use mathematical objects called matrices and vectors. Dirac [1958] reasoned that we need:

“...mathematical quantities that can be added together to give quantities of the same kind. The most obvious of such quantities are vectors.”

Let’s now get these mathematical concepts in our tool chest.

Mathematics: matrices (singular: matrix)

A matrix is an array (rows and columns) of numbers. The number of rows and columns need not be the same, but a matrix cannot have a “hole”, an element that is not defined. Here is an example of matrix with three rows and three columns:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (2.1)$$

Formalism, here meaning the way I communicate with type, is necessary, in order to keep things straight. Since they will get rather complicated as we go on further, I here will typeset all matrices with bold, sans serif type, like **A**.

Matrix addition

We can add matrices together by adding each element to its analogous element. If

$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \quad (2.2)$$

then

$$\mathbf{B} + \mathbf{C} = \begin{bmatrix} (2 + -1) & (1 + 1) & (0 + 3) \\ (3 + 0) & (0 + 4) & (1 + 7) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 8 \end{bmatrix} \quad (2.3)$$

Note that the addition of matrices is commutative, i.e. the answer does not depend on the order:

$$\mathbf{B} + \mathbf{C} = \mathbf{C} + \mathbf{B} \quad (2.4)$$

Matrix multiplication

We can also multiply matrices together

(I am going to introduce vectors, and then we come will look at matrix multiplication)

Vectors

A vector is a list of numbers. For instance $\mathbf{x} = (1, 2, 0, 3)$ is a (row) vector. Vectors are not the same mathematical beasts that matrices are, although you can add matrices

⁹“The neurons of the brain do nothing more than vector calculation.” ?, 56. Translated by the author.

Operators
are best
represented
mathemat-
ically by
matrices

and vectors together and you can multiply matrices and vectors together. Adding vectors together is easy, you just add element by element, similar to matrix addition.

There are different ways to multiply vectors together, you have to decide exactly how this is to be done and then define it, and name it. We will be concerned with what is called the *dot* or *inner product* of vectors. It is defined as

To add and multiply vectors and matrixes together, we can think of a vector as being a single column matrix. An example of a vector written as a list, or a one-column matrix is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \quad (2.5)$$

Vectors can be use to describe a quantity that has a magnitude (size, length) and a direction, for instance velocity. If something is moving, it has a certain speed and a certain direction. You cannot move something in physical space without it having a certain speed and a certain direction. Vectors help to describe such motion.

For example, a one-dimensional vector is a number with a direction. The vector 5 can be thought of as an arrow with the length 5 lying flat on a number line. A two-dimensional vector, say the vector (5,3) is an arrow pointing in the direction 5 units. I am on purpose not showing you a graph with a vector here. The reason is that in quantum mechanics, a vector is essentially something quite different although mathematically described in the same way. We will shortly come to vectors in quantum mechanics. There orientation is important, but position in space is not so important and is defined first when a coordinate system is introduced.

Why do we need vectors in quantum mechanics if position is not important? Good question! I cannot answer that in complete depth (yet), I think you will shortly see why. The first-level answer is that QM needs lists of numbers. In fact, in QM these are infinite lists (why must they be *infinite*? See later). QM also needs to add, subtract, and above all, multiply these lists together. Things soon get into vectors of more dimensions (meaning more rows and columns) and these are called matrices. There turns out to be a mathematical apparatus for handling vectors and matrices called linear algebra. It turned out to be perfect for QM. The use of vectors and matrices has proved crucial in the modern formulation of QM.

To remind ourselves that in QM we are dealing with special vectors and not just your run-of-the-mill velocity vectors, Dirac introduced a special name for vectors that are connected with states of a system. These are called (first by Dirac) as *ket vectors* and *bra vectors* or just kets” and bras.¹⁰

The symbol for a general ket is $|\ \rangle$. It is a kind of a place holder to put something in that implies a specific calculation method.¹¹ If we want to specify a particular ket, say ket A, then the A is put between a vertical line and an angular line bent to the right:

¹⁰Evidently Dirac was not disturbed by the similarity of this term with an abbreviation for a particularly feminine piece of clothing. I had asked in the introduction: *Where is the feminine in QM?*

¹¹In the sense that Wheeler, Thorne and Misner call tensors, computational machines, see *Gravitation*, Charles W. Misner [1970].

This is not actually how QM began, in the Appendix there will be a short history of QM.

$|A\rangle$. Kets can be multiplied by numbers (in general, complex numbers) to give other kets: $5|A\rangle = |5A\rangle$. Also they can be added together to give other kets:

$$|A\rangle + |B\rangle = |A + B\rangle.$$

Kets can also be multiplied together to give other kets:

$$|A\rangle \times |B\rangle = |AB\rangle.$$

But watch out!

$$|AB\rangle \neq |BA\rangle.$$

If A and B are matrices, then in general the product $AB \neq BA$. In QM, this has far-reaching consequences. It is one of those things: the introduction of a formalism (linear algebra) happens to correspond to a deep-reaching physical interpretation. Why is this so? I do not know. We will be come back to this later. For the moment, just note that this means the matrices (one-dimensional vectors) A and B do not commute. The mathematics of QM is non-commutative. If you want to dive right now into more on this, see the appendix entry [non-commutivity](#). I am going to continue plodding along here.

Until now, I have only mentioned kets, what about bras? A bra looks like a mirror-image of a ket:

$$\langle \quad |$$

If we want to specify a particular bra, say bra B, then the B is put between an angular line bent left and a vertical line:

$$\langle B|$$

Of course, this is all formalism. I have not said anything about what a bra *is*! The reason why a bra is put between an angular line bent left and a ket between an angular line bent right will become clear shortly. For the moment, the left bracket simply means: the vector you put in here is the complex conjugate of some particular ket. For details of complex numbers and a complex conjugate, see the next section. In (I hope) simple terms, the complex conjugate is a number related to another number in a specific special way.

2.3.2 Numbers and complex numbers

For a deeper understanding of QM (and that is what I want to give you here) I need to make a digression into complex numbers.

Numbers are extremely important in QM. Numbers mediate between QM as an intellectual system and the real, material world. There are particular numbers that make all the difference, like Planck's constant, h , which stands for a very particular unit of energy, which will be described later. There are four different kinds of numbers that mathematicians deal with (not three or five, but exactly four!):

integers are 1, 2, 3, 4, ... (0 is a special integer and -1, -2, -3 are negative integers).

Actually, integers are used in QM (as in other parts of physics) in two ways. One is in their ordering function, like m_1, m_2, m_3 in which the subindices 1, 2, 3 denote an

ordered series: m_3 comes after m_2 . Try to make an order in which “2” comes after “1”. It does not work! No matter what language you speak, two, in that language, comes after one and before three. Why is this so? Think about that. . .

Another function of integers is that they denote a *value* (this is called their “cardinal” function): five apples are *more* than four apples, $m^2 = m \times m$.

real numbers like 1.00, -2.6, 5.2587

rational numbers 1.313131313131... These are equivalents of some fraction, $1/3 = 0.3333333$ (unending row of 3's) $\sqrt{2} = 1.4141414$ (repeats indefinitely)

irrational numbers like π , which stands for the ration of a circle to its diameter and always equals 3.141516 (never repeating, never ending).

complex, imaginary numbers like $3+2i$. This i is the same i as we met above in Table 1.1 (the fundamental QM equations). QM uses all the kinds of numbers listed above, but in particular, complex numbers. QM can not function without representing its entities as complex numbers. So we will have to remind ourselves what these things are.

2.3.3 Complex numbers

Complex numbers hold a special meaning in QM and therefore I will describe them in a bit more detail. Complex numbers arrived in the field of mathematics through arabian mathematicians in 16?? when it was found that quadratic equations¹² could not be solved with real, rational, and irrational numbers. For instance, the equation

$$x^2 = -1$$

could not be solved since there was no way to conceive of the square root of a negative number. Either mathematicians gave up or got creative. They created the number $i = \sqrt{-1}$ to represent the square root of negative one. i was called the *imaginary*¹³ unit. you can multiply any number by i and that makes it an imaginary number. With i and imaginary numbers, the mathematicians could now solve any quadratic equation, theoretically at least.

However, the story does not end here. By combining a real, rational, irrational number with i and adding this to another real, rational, irrational number, you have a system of numbers that satisfy all the normal rules of combining, multiplying, dividing numbers. Let's practice. The complex number $3 + 2i = 3 + 2\sqrt{-1}$. To simplify this a bit, mathematicians devised a shorthand notation. Let's agree that when we put a comma between

¹²Quadratic equations arrived when calculating the surface of a field. or a square. An example is $x^2 + 8x - 1 = 0$. In a quadratic equation, at least one unknown (like x) is squared. See more on equations in the Appendix.

¹³This name for i was first introduced by in 16???. That it is called “imaginary” is unfortunate, since it exists in mathematics, it is just hard to conceive in terms of our physical world. Another example of how QM forces us to broadening and define our concepts more precisely. And that this, amazingly enough, effects how we think about the world, and can effect how we interact with that world. At least intellectually for the moment. (But when we accept that what we physically do is obviously just as affected, the ball comes home.)

3 and $2i$ and enclose this “ordered pair” in braces, that it means the first number (left of the comma) is a real, rational, or irrational number; and the number to the right of the comma is a real, rational, or irrational number times i . We just will not write the i , but we know it is there. We agree that we will always mean this, what is on the left is real, what is on the right is imaginary. We have therefore created an ordered pair. That is, we write

$$(3 + 2i) \text{ as } (3, 2)$$

and these two mean exactly the same thing, we have just simplified the writing a bit. Now you can add, subtract, multiply and divide any number as an ordered pair. That is, you apply the rule $+$ (= addition) for instance, first to the real part and then to the imaginary part separately:

$$(3, 2) + (5, 4) = (3 + 5, 2 + 4) = (8, 6)$$

Remember that

$$(8, 6) = 8 + 6i = 8 + 6\sqrt{-1} = 8 + \sqrt{-6}$$

Without complex number this is not possible, with complex numbers, you need no other kind of number. With complex numbers, the whole spectrum or world of numbers is complete!

2.3.4 Some other special numbers in QM

h = Planck’s constant. This stands for an incredibly small unit of energy, 6.626×10^{-34} Joules or

= the ratio of color (energy) E of a particle/wave (a photon, for instance) to its frequency ν , and in turn wavelength λ is related to frequency ν and the speed of light c . Mathematically:

$$E = h\nu = \frac{hc}{\lambda} \quad (2.6)$$

The de Broglie equations relate the wavelength λ and frequency ν to the momentum p and energy E , respectively, as

$$\lambda = \frac{h}{p} \quad \text{and} \quad f = \frac{E}{h} \quad (2.7)$$

where h is Planck’s constant. The two equations are often written as

$$p = \hbar k \quad \text{and} \quad E = \hbar\omega \quad (2.8)$$

\hbar pronounced “h-bar” $\hbar = h/\pi$. This combination is so frequent in QM that it got its own designation. is a combination of three numbers: h = Planck’s constant where $\hbar = h/(2\pi)$ is the reduced Planck’s constant (also known as Dirac’s constant, pronounced “h-bar”), k is the angular wavenumber, and ω is the angular frequency.

Using results from special relativity, the equations can be written as

$$\lambda = \frac{h}{\gamma mv} = \frac{h}{mv} \sqrt{1 - \frac{v^2}{c^2}} \quad (2.9)$$

and

$$f = \frac{\gamma mc^2}{h} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{mc^2}{h} \quad (2.10)$$

where m is the particle's rest mass, v is the particle's velocity, γ is the Lorentz factor, and c is the speed of light in a vacuum.

$$\nu = \frac{\gamma mc^2}{h} \quad (2.11)$$

where m is the particle's rest mass, v is the particle's velocity, γ is the relativistic correction (Lorentz factor), and c is the speed of light in a vacuum.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2.12)$$

See the article on group velocity for detail on the argument and derivation of the de Broglie relations. Group velocity (equal to the electron's speed) should not be confused with phase velocity (equal to the product of the particle's frequency and its wavelength). Planck's constant has dimensions of energy multiplied by time, which are also the dimensions of *action*. In SI units, the Planck constant is expressed in joule seconds = 6.6261×10^{-34} Js, a very small number. The smaller the wavelength, the bigger the momentum of the corresponding particle, making electrons, with their smallness and correspondingly small momentum, one of the most "wavelike" of the particles.

2 The number 2 (exactly 2, that is both ordinal and cardinal).

Pi $\pi = 3.1416 \dots$ is the (constant) ratio of the circumference to the radius of a circle. Pi is never repeating, never ending. It has something to do with the essence of circles. The reduced Planck constant is used when frequency is expressed in terms of radians per second instead of cycles per second. The expression of a frequency in radians per second is often called angular frequency ω , where $\omega = 2\pi\nu$.

→ **Here ends the edited text. Below is incomplete but actively (see version date on title page) undergoing work!**

In QM, kets and bras are vectors that represent lists of numbers, As I wrote, kets and bras are particular vectors in QM.

The great thing about Dirac's notation – once you get it – is the simplicity with which it can be used to describe complicated processes. I am now going to give the classical example (known by thousands of physics student the world over), a lecture by Richard Feynman ([Feynman et al., 1965b] III-1).

2.4 Feynman's double-slit experiment

Now that we have learned the basics of Dirac's bras and kets, let's apply them to an experimental situation. Bear with me, it is worth it. We are going directly to the heart of QM.

Preliminary remarks

Dirac (slightly rephrased): To be practical, physics has to limit itself to things that can be observed. Observation is only possible when the object to be observed interacts with its environment. Causality can only apply to a system that is not disturbed [Dirac, 1958] (p. 3-4). Since you disturb an object in the act of observing it, an unavoidable indeterminacy is introduced (another reason why QM is non-casual).¹⁴ Therefore, in general, one can only calculate the probability of obtaining any particular result.

This necessary introduction of probability has nothing to do with statistics in the classical sense, but with the fact that we are dealing with interaction of an act of observation (by a human – the one who is asking the question and making the theory) upon an object. Probability in basic quantum mechanics does not have to do with statistics of many things; the statistical nature (of the result of an observed interaction) may refer to many particles in classical physics, however it refers to one particle in quantum mechanics. Or, put another way, statistics in classical physics refers to what we can say about *collective nature* and in quantum mechanics refers to what we can say about *one single particle*. Is the statistical nature of the QM formulation intrinsic in nature, or is it due to the fact that a (conscious) observer must make a measurement to get a result (an answer to a conscious question)? Some people think so, others not. For instance there is evidence that processes in nature are statistical when they are not the result of an observed measurement. But how do I know this?

The amplitude picture

Imagine that a physicist wants to know something about a beam of light. What happens when we shine light on a screen?

1. Probability amplitude

The probability that a particle will arrive at X when let out of a source S can be represented quantitatively by the absolute square of a complex (we now know what this is!) number called a probability amplitude:

$$\langle \text{particle arrives at } x \mid \text{particle leaves } S \rangle \quad (2.13)$$

or, simply (here comes pure Dirac – keep in mind we are abbreviating many things, both mathematics and QM philosophy here at once)

$$\langle x|S \rangle \quad (2.14)$$

2. Sum of amplitudes

When a particle can, that is, theoretically you can measure it, is possible, reach a given state by two possible routes, the total amplitude for the process is the sum of the amplitudes for the two routes considered separately:

$$\langle X|S \rangle_{\text{complete process}} = \langle x|S \rangle_{\text{through route 1}} + \langle x|S \rangle_{\text{through route 2}} \quad (2.15)$$

or (Dirac)

$$\langle X| \rangle_{\text{complete process}} = \langle x|S \rangle + \langle x|S \rangle \quad (2.16)$$

¹⁴See also Wheeler in [Tho, 2009]

3. Product of amplitudes

When a particle goes by some particular route, the amplitude for that route can be written as the product of the amplitude to go part of the way $\langle X \rightarrow 1|$ with the amplitude to go the rest $\langle 1 \rightarrow S|$ of the way:

$$\langle X|S \rangle_{\text{complete route}} = \langle X|1 \rangle + \langle 1|S \rangle \quad (2.17)$$

If events occur in succession (first process 1 then process 2), then the total is the sum of the individual products:

$$\langle X|S \rangle_{\text{both together}} = \langle X|1 \rangle \langle 1|S \rangle + \langle X|2 \rangle \langle 2|S \rangle \quad (2.18)$$

In this way you can build up the calculation of probability amplitudes for more complicated processes.

These statements have far-reaching consequences: We are here at heart of QM, and what makes it such a special theory. To make sure this is clear, let me repeat the above in a slightly different formulation, quoting again from Feynman (Lectures III, 1-10; with some additions for clarity).

4. Probability

The probability of an event in an ideal experiment¹⁵ is given by the square of the absolute value of a complex number ϕ which is called the probability amplitude:

$$\begin{aligned} P &= \text{probability} \\ \phi &= \text{probability amplitude} \\ P &= |\phi|^2 \end{aligned}$$

5. Possibilities

If two routes are possible, that is, if an event can occur in several alternate ways, the amplitudes for [the result of an experiment in which one cannot determine which of the routes was actually taken, then the measured outcome of] the event is the sum of the amplitudes for each route considered separately. There is interference [and we are measuring waves].

$$\begin{aligned} \phi &= \phi_1 + \phi_2 \\ P &= |\phi_1 + \phi_2|^2 \end{aligned}$$

6. Performing the experiment \rightarrow collapse of wavefunction

If an experiment is performed which is capable of determining whether one or another alternative is actually taken, the probability for the event is the sum of the probabilities for each alternative. The interference is lost [and we are now measuring particles!].

$$P = P_1 + P_2 = |\phi_1|^2 + |\phi_2|^2$$

¹⁵“An *ideal experiment* is one in which all of the initial and final conditions of the experiment are completely specified.” In other words, “there are no uncertain external influences, no other things going on that we cannot take into account.” Feynman et al. [1965b]1-10.

2.5 Discussion of Feynman's example

Note the difference: in the second case it is *possible* to measure the alternatives and then there is no interference pattern. This appears to say that just the theoretical possibility that we can measure something, already changes its state! As Feynman puts it: quantum mechanics forces physicists to think in a special way in order to avoid getting into inconsistencies. If you are capable of determining whether route 1 or route 2 was taken by some particle (i.e. you have an apparatus that is capable of differentiating and you choose to run an experiment to make the differentiation), then at the end of the experiment, you could say whether the particle went by way of route 1 or route 2. But if you do not try to tell which path it took [i.e. you do not have an apparatus capable of measuring which path, or you do but you choose to make another experiment using another apparatus], then you may not say that it goes by way route 1 or route 2. [You must say that it went both ways, like a wave]. If you do anyway think you can determine which route and you try and start to make deductions based on this, you will make errors in the analysis [Feynman et al., 1965b, 1-9]).

Why this odd situation? (Does a conscious observer, or the consciousness of the observer, or the unconscious itself play a role here?) In the quantum literature this is called the problem of pure and mixed states, in popular literature: the wave/particle duality, and: "Is Schrödinger's cat dead or alive?" It is related to the problem often referred to as *the collapse of the wave function* or *reduction of the wave packet*. These are discussed in the section on the wave function picture.

This odd situation gets even worse if you consider what happens if you decide to measure which route *after* the particle actually took one or the other, or both, routes. Clearly your decision cannot influence the past motion of something. Or can it? A good macroscopic example of this situation is given by J. A. Wheeler¹⁶ with light particles that actually went one or the other (or both) paths more than 50,000 years ago, and arrived and were observed at Mount Palomar Observatory in 1983. Due to the observational setup, the light particles must have travelled both paths. But these two alternate paths were separated by several light-years (several millions of kilometers)! How is this possible? Not only this, but if a decision now seems to effect something so far in the past, then maybe a decision by somebody in the future is affecting things that happen now... I will discuss this later. Is this science or science fiction!

Feynman writes:

"One might like to ask: How does it work? What is the machinery behind the law? No one has found any machinery behind the law. No one can 'explain' any more than we have just 'explained'. No one will give you any deeper representation of the situation. We have no ideas about a more basic mechanism from which these results can be deduced.

2. [Check this] Complicated process can be built up on the basis of adding amplitudes for separate processes (this is called *superposition of states* and is also can not taken for granted at the outset. It makes the present quantum theory a 'linear theory') and also has consequences for our understanding of matter (or the psyche, or both).

¹⁶Wheeler ...

The wave function (ψ) picture: The quantum mechanics usually taught in physics and chemistry classes emphasizes the wave function description and the Schrödinger equation. The wave function does not have anything to do with a physical wave. Schrödinger did not derive his equation, but pulled it out of the blue with help from intuition, analogy, and knowledge of physics.¹⁷ In the amplitude formulation, the amplitude contains the specific numerical value that will appear on your measuring apparatus. Whereas the wave function contains all the information about the object being observed, but since you can only observe one value of an observable at a time, you only see those parts of the wave function that have been reduced during one specific measurement. The actual observation is thus a reduction of all that information to one, namely the value of the observable being measured. Obviously, as you are probably thinking, not everything can be observed. Correct. This is another wonderful and frustrating (for the physicists who want to know everything) part of quantum physics. But that is another story.

As M. Jammer puts it: the wave function:

...does not itself represent a course of events in space and time. It rather expresses our knowledge of events. ...what the physicist does is essentially this: from an observation he constructs a wave function which obeys in its progress the laws of quantum mechanics but constitutes at any time merely a catalogue of probabilities for the results of subsequent measurements or observations. The state of affairs between one observation and the next cannot be described; but how the observation is described depends upon the experimental setup chosen by the experimenter. The results of observations thus can never completely be objectified.

...the language of quantum mechanics is a language of interaction and not of attributes: processes, and not properties, are the elements of its syntax.¹⁸

Some writers go so far as to call quantum mechanics *process* physics.¹⁹

Formulation 2. As opposed to the above, relatively abstract and thoroughly thought-out, ripe formulation of traditional quantum mechanics, let's go back to the beginning of quantum mechanics. To get a feeling for the way in which physicists looked at the matter and their attitude. This will help set the background for interpreting the responses from the unconscious (the formulation of the theories).

¹⁷See ...

¹⁸M. Jammer, 199....

¹⁹citation needed here!

→ this is QM03-waveequation.tex

2.6 What is the wave equation about?

The wave equation is an equation of the species *differential* equations. It is important to understand what these are. So let's first look at non-differential equations. I am only going to give a few examples of the whole world of equations

2.7 Equations and differential equations

Well, what is in equation, actually? It is a shorthand way to state relationships.

2.7.1 Scalar equations

A *scalar* equation states that what is on the left side *always* has and always will equal what is on the right side.

Example: The length of the diagonal (D) of a rectangle is equal to the square root of the sum of the squares of the lengths of the two adjacent sides (let's call these lengths A and B). Now that is a lot of words! Look at how we can write this as an equation:

$$D = \sqrt{A^2 + B^2} \quad (2.19)$$

That is pretty simple! As long as we agree on what D and A and B mean and our measuring system is moving at the same speed, this equation is independent of time and no matter when we measure the sides, the square of the sum of there squares will always be equal to the diagonal.²⁰

2.7.2 Quadratic equations

Quadratic equations are important to understand why complex numbers are needed.

...

Omygod, I am getting totally stuck. I wanted to get my BIG POINT out on paper (04.09.2010):

- The wave equation is a differential equation.
- It says: "If you want to know what something is doing then (at another time), then stick it in here."
- In this sense it is a complex computational machine.
- Maybe this is how our brains work. They do calculate like matrixes²¹ - Now what I would like to do in this paper, is to show a few graphic plots of what the wave equation does with eveloves, just to get an idea of what it is about.

²⁰According to the General Theory of Relativity, lengths depend on the system in which they are measured and the speed this system is traveling.

²¹"The neurons of the brain do nothing more than vector calculation." ?, 56. Translated by the author.

- It is a standing wave, to infinity that does not change.
- What does not change?
- An envelope of possibilities?
- The envelope evolves?

3 Appendix

Appendix

Dirac Brackets The name *bracket* is taken from the *Poisson bracket*, a mathematical operator in Hamiltonian (classical) mechanics. A Dirac bracket formulates the inner product of two vectors, which give a number. Although this has meaning for any two vectors, in quantum mechanics the state of something measurable (a photon, for instance) can be represented by a vector.

The inner product (or dot product) of two states is denoted by a bracket, $\langle\phi|\psi\rangle$, consisting of a left part, $\langle\phi|$, called the bra, and a right part, $|\psi\rangle$, called the ket.

Bras and kets are infinite vectors and the mathematical space in which they exist is called a *Hilbert space*.

Dirac represented an *operator* in quantum mechanics as bras and kets. He formalized them as being enclosed by a straight line and a left or right bracket sign. If A and B are vectors then $\langle B|$ is a bra and $|A\rangle$ is a ket. The scalar product of two vectors (as written by Dirac) then takes the form:

$$\langle B|A\rangle \quad (3.1)$$

Bras and Kets Dirac in his genius way took these from the term *Poisson bracket*, a mathematical operator in Hamiltonian (classical) mechanics. In quantum mechanics, bras and kets stand for states of a system called state vectors. A Dirac bracket formulates the inner product of two state vectors, which give a number. The numbers in the list making up each state vector (bra or ket) represent all the possible states that a particular system can ever find itself in (as you can imagine, this is an infinite list). Putting in vectors and using vector multiplication to calculate the bracket (taking the inner product) yields a number, the particular state that one particular variable finds itself in. Numerically, bras are complex conjugate (imaginary conjugate) of kets. Thus they are related, but not the same. That they are not the same is a subtle but important point. What does this mean?

Vectors A vector is a list of numbers, in the case of bras and kets, lists of complex numbers. A complex number has a *real* part and an *imaginary* part (in the sense of being a multiple of the square root of negative one, something that is not a "real" number). Bras are the complex conjugate (the imaginary conjugate) of the kets. Thus they are related, but not the same.

The inner product (or dot product) of two states is denoted by a bracket, $\langle\phi|\psi\rangle$, consisting of a left part, $\langle\phi|$, called the bra, and a right part, $|\psi\rangle$, called the ket.

Dirac represented an operator in quantum mechanics as bras and kets. He formalized them as being enclosed by a straight line and a left or right bracket sign. If A and B are vectors then $\langle B|$ is a bra and $|B\rangle$ is a ket. The scalar product of two vectors (as written by Dirac) then takes the form:

$$\langle B|A\rangle \quad (3.2)$$

In Dirac's formulation, bras and kets stand for states of a system, state vectors. The numbers in the list making up each state vector (bra or ket) represent all the

possible states that a particular system could find itself in. Bras and kets are infinite vectors and the mathematical space in which they exist is called a *Hilbert space*. In order to determine, that is to observe (I am tempted to say “be consciously aware of a particular state,” but this is not mainline quantum mechanics), these state vectors have to be operated on by some measuring system and some quantity has to be measured. (John Wheeler used to say that measurement *creates* the object being measured.²²) The basic mathematical formulation of this statement is the (one-dimensional) quantum mechanical wave equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \quad (3.3)$$

\hat{H} is the *Hamiltonian*, the total energy of the system. The Hamiltonian is a linear operator whose action (operation) here is to twice differentiate the wavefunction ψ with respect to the spatial coordinates. (In essence it asks what is the rate of change of the velocity (that is, the acceleration) of the wavefunction. This equation is a kind of mathematical “machine” that can be used to calculate the probability amplitude ψ that a given state developing in time (t) and space (\mathbf{x} = three coordinates). \hat{H} takes as input one ψ and produces another in a linear way, a function-space version of a matrix multiplying a vector.

This equation is called a *wave equation* because it describes how the more something *changes* in time (accelerates or decelerates), the more it spreads out (or condenses) in space. This is just how a wave moves, it builds up and then settles down again. Actually the equation describes a *standing* wave, a wave that just travels through space and itself does not change. What is changing here in time and space? The space is not our familiar three-dimensional space, but actually a mathematical space (in this case a Hilbert space filled with Hermitian matrixes, see [Appendix](#)). The *probability* that something will find itself in a particular state; the wave is a wave of probability! “Here, at this time, it is more likely to find itself in a particular state; there, at that place, it is less likely to find itself in that particular state.”

In the formulation with bras and kets, if $|\psi(t)\rangle$ is a state of the system at time t , then

$$H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \quad (3.4)$$

This is the simplest form of the Schrödinger equation in Dirac's formulation. As an aside, note from the left side of this equation that i and \hbar are part of an operator. When numbers act as multipliers, they are operators!²³ The wave function

²²He also said that “No phenomenon is a phenomenon until it is a recorded phenomenon.” [Tho, 2009]

²³<http://en.wikipedia.org/wiki/Schrödinger> “Discouraged, he put away his calculations and secluded himself in an isolated mountain cabin with a lover. While there, Schrödinger decided that the earlier nonrelativistic calculations were novel enough to publish, and decided to leave off the problem of relativistic corrections for the future. He put together his wave equation and the spectral analysis of hydrogen in a paper in 1926.” For completeness the time-dependent Schrödinger equation $-\hbar^2/8\pi^2m\partial^2/\partial x^2 + V = 2\pi^2E/\hbar^2$. This equation has the structure of a diffusion equation with an imaginary diffusion coefficient (the i). Diffusion equations say that the steeper the gradient, the faster something runs (or diffuses) away. Take a hill, the higher the hill, the more material is eroded off of it in a rainstorm; or heat: the hotter a glass the more rapidly it cools... (The equations does not tell you how this happens, only the quantitative relationships between the things being represented.) Chemists note that you cannot calculate anything with the matrix or the Feynman approach, but you can do a remarkable amount with the Schrödinger equation. They are right. On the other hand,

contains all the information about the object being observed. However, since you can only observe one value of an observable at a time, you only see those parts of the wave function that have been reduced during one specific measurement. In effect, what we experience (what we can become conscious of) is the reduction or collapse of the wave function. In Dirac's formulation, the scalar product (a rule for multiplying two vectors together) of a bra and ket $\langle A|B\rangle$ is the probability amplitude for the state $\langle A|$ to collapse into the state $|B\rangle$.

Let's try to write this out more clearly. the bras and kets are vectors *and* linear operators. In the Schrödinger equation in Dirac's formulation this means we

What does this mean? When you solve this equation for the Hamiltonian, the total energy, for instance, (the "collapse"), you get the probability that the system you measure is in the state $\langle A|$. This is as close to "understanding" matter that physics will ever get us. In other words, what can become conscious. It does not say anything else about anything else.

Hermitian matrices A Hermitian matrix (or self-adjoint matrix) is a square matrix with complex entries which is equal to its own conjugate transpose that is, the element in the i th row and j th column is equal to the complex conjugate of the element in the j th row and i th column, for all indices i and j ²⁴. Since H Hamiltonian is a Hermitian operator (i.e, written as a Hermitian matrix) the energy is always a real number.

Hamiltonian In quantum mechanics, the Hamiltonian H is the observable corresponding to the total energy of the system. -explain observables and operators - I thought the \hat{H} is an operator that takes the Laplacian...

Hamiltonian as an observable It is a Hermitian matrix, that, when multiplied by the column vector representing the state of the system, gives a vector representing the total energy of the system. As with all observables, the spectrum of the Hamiltonian is the set of possible outcomes when one measures the total energy of a system.

The eigenkets (eigenvectors) of H , denoted by $|a\rangle$ in Dirac's bra-ket notation, provide an orthonormal basis for the Hilbert space. The spectrum of allowed energy levels of the system is given by the set of eigenvalues, denoted E_a , solving the equation:

$$H |a\rangle = E_a |a\rangle \quad (3.5)$$

Non-commutivity, non-commuting algebra

This means that changing the order of what you do changes the result. For instance, if you dry your clothes first and then wash them, you get a significantly dif-

you do not really think about what it is you are calculating. The wave equation has as many philosophical problems as any other formulation of quantum mechanics. All these different formulations are equivalent. Equivalently odd. They are eigenvalue equations. (Which means that ...) Another representation would be to say that if you operate on the wave function, you get a multiple of ...? states, only one of which at each instance of time is observable (? guessing at what I meant, January 2006).

²⁴

For instance, the matrix $\begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix}$ is a Hermitian matrix.

ferent result than if you wash first and then dry. (In the first instance, your clothes are washed but wet.) That is, washing and drying are non-commutative operations.

Matrix multiplication is non-commutative. When Heisenberg first realized that his new formulation of quantum mechanics was the same as matrix multiplication, he was disturbed that this introduced non-commutativity. (Subtraction and division are also non-commutative operations, but these do not have the far-reaching consequences as multiplication in quantum mechanics).

In the mathematical sense, the strangeness of quantum mechanics emerges from the simple fact that the quantum Poisson bracket is non-commutative.

Some History of QM Werner Heisenberg published in 1925 his own method for handling these things before he realized the connection to linear algebra. Soon after Max Born and Pascal Jordan made the connection 1926 and later (still in 1926) with Heisenberg published the “three man paper” [?]. In the meantime (early 1926), Paul Dirac had also made the connection and developed the whole notation presented here [Dirac, 1958]. Here is the link to the wikipedia article on Heisenberg non-commuting matrices Wikipedia [2009] http://en.wikipedia.org/wiki/Introduction_to_quantum_mechanics#Schr.C3.B6dinger_wave_equation

Some history of how non-commutivity got into physics is reported in: Bernstein [2005].

Laplacian The Laplacian is the divergence of the gradient. In other words, just how does the gradient change (for example, in time or space)? Does it get steeper or flatter.

Divergence The divergence is an operator that measures the magnitude of a vector field's source or sink at a given point; the divergence of a vector field is a (signed) scalar. For example, consider air as it is heated or cooled. The relevant vector field for this example is the velocity of the moving air at a point. If air is heated in a region it will expand in all directions such that the velocity field points outward from that region. Therefore the divergence of the velocity field in that region would have a positive value, as the region is a source. If the air cools and contracts, the divergence is negative and the region is called a sink. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. In physical terms, the divergence of a three dimensional vector field is the extent to which the vector field flow behaves like a source or a sink at a given point. It is a local measure of its *outgoingness* — the extent to which there is more exiting an infinitesimal region of space than entering it. If the divergence is nonzero at some point then there must be a source or sink at that position 1. (Note that we are imagining the vector field to be like the velocity vector field of a fluid (in motion) when we use the terms flow, sink and so on.)

Gradient the gradient of a scalar field is a vector field which points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change. For instance, consider a room in which the temperature is given by a scalar field T , so at each point (x, y, z) the temperature is $T(x, y, z)$ (we will assume that the temperature does not change in time). Then, at each point in the room, the gradient of T at that point will show the direction in which the

temperature rises most quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

Consider a hill whose height above sea level at a point (x, y) is $H(x, y)$. The gradient of H at a point is a vector pointing in the direction of the steepest slope or grade at that point. The steepness of the slope at that point is given by the magnitude of the gradient vector.

There are numerous mathematically equivalent formulations of quantum mechanics. One of the oldest and most commonly used formulations is the transformation theory proposed by Cambridge theoretical physicist Paul Dirac, which unifies and generalizes the two earliest formulations of quantum mechanics, matrix mechanics (invented by Werner Heisenberg [?], [?]) and wave mechanics (invented by Erwin Schrödinger [?]).

In this formulation, the instantaneous state of a quantum system encodes the probabilities of its measurable properties, or observables. Examples of observables include energy, position, momentum, and angular momentum. Observables can be either continuous (e.g., the position of a particle) or discrete (e.g., the energy of an electron bound to a hydrogen atom).

The term transformation theory refers to a procedure used by Dirac in his early formulation of quantum theory, from around 1927 [?].

The term is related to the famous wave-particle duality, according to which a particle (a *very small* physical object) may display either particle or wave aspects, depending on the observational situation. Or, indeed, a variety of intermediate aspects, as the situation demands. This transformation idea also refers to the changes a physical object may undergo in the course of time, whereby it may move between positions in its Hilbert space.

The mathematical concept of a Hilbert space, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra from the two-dimensional plane and three-dimensional space to infinite-dimensional spaces. In more formal terms, a Hilbert space is an inner product space — an abstract vector space in which distances and angles can be measured — which is *complete*, meaning that if a sequence of vectors is Cauchy, then it converges to some limit within the space.

Equations Egyptian mathematicians knew that the length of a diagonal of a rectangle was equal to the square root of the two sides. If a is the length of one side and b the length of the other side and d is the length of the diagonal, then

$$d = \sqrt{a^2 + b^2}$$

This is an equation. Equations in mathematics are statements that something on the left side equals something on the right side of an equal sign. Normally when there is an equals sign, the two sides are equal. theoretically, as we have defined the formalism this is true. But practically, to really set them equal, we have to solve the equation. This is not always so easy as writing the equation down. In fact, there are many, infinite many, equations that can not be solved exactly. This is most of QM! For these we have to make approximations, although the formalism is

an “exact” statement of what *is*. Is it what it is or is it not?

To solve an equation you need to know what everything means. We are at the end and at the beginning!

QM Outtakes

from: John Wheeler, relativity, and quantum information Charles W. Misner, Kip S. Thorne, and Wojciech H. Zurek *Physics Today*, April 2009, 40-46

Wheeler's last blackboard John Wheeler taught a two-year course on quantum measurement at the University of Texas at Austin, in 1977-79. In the course's final class, according to notes taken at the time by one of us (Zurek), Wheeler wrote the following list of ideas and then discussed them:

1. We don't understand how the universe came into being.
2. We will first understand how simple is the universe when we recognize how strange it is.
3. When we understand how it came into being, it will seem so compelling that we will all say how stupid we have been.
4. Therefore, we can afford many mistakes in the search. The main thing is to make them as fast as possible.
5. No explanation is an explanation that does not explain how the universe comes into being out of nothingness; not out of the vacuum of physics with its fluctuations and virtual particles, but out of nothingness. No laws, no particles, nothing.
6. *Omnibus ex nihil ducendis sufficit unum.* (One principle suffices to obtain everything from nothing.)
7. No principle is more appealing for this purpose than the principle that many a game is not a game until the line is drawn across the empty courtyard: complementarity and the distinction between observer and system observed.
8. Physics has to give up its impossible ideal of a proud unbending immutability and adopt the more modest mutability of its sister sciences, biology and geology.
9. If the kingdom of life and the highest mountain ranges are brought into being by the accumulation of multitudes of small individual processes, it is difficult to see what else can give rise to the universe itself.
10. What other possibility is there for law without law except the statistics of large numbers of lawless events?
11. No elementary process is as attractive for this statistics as the elementary act of observer-participatorship.
12. The quantum theory of fluctuations of geometry tells us that the concepts of before and after lose all application at distances of order the Planck length or less. If the concept of time fails anywhere, it must fail everywhere.
13. Time is not a primary category, and the asymmetry of time between past and future is not a primary category in the description of nature. It is secondary and derived.
14. The elementary act of observer-participatorship transcends the category of time (delayed-choice double slit).

15. No working picture that can be offered today is so attractive as this: the universe brought into being by acts of observer- participatorship; the observerparticipator brought into being by the universe (self-excited circuit).
16. The laws of physics reveal as little about the deeper structure of the universe as the laws of elasticity reveal about the quantum mechanics of the solid state. Symmetry principles summarize law but also hide machinery behind the law.
17. Philosophy is too important to be left to the philosophers.

Same source: The passive observer of Newtons classical universe becomes, in our quantum world, a participator. The participators selection of what to measure determines the set of possible outcomes. When the preexisting quantum state is not one of those possible outcomes, it is doomed: The measured system will jump into one of the possible outcomes, with a probability given by Max Borns famous rule, $p_k = |\Psi_k|^2$. In effect, as Wheeler saw it, the wavefunction of the universe was reset in the process. So in our quantum world, the future is determined in part by the questions posed by observer participators, and by measurement-induced random quantum jumps. “No phenomenon is a phenomenon until it is a recorded phenomenon”- was Wheelers pithy summary of Bohrs similar viewpoint, and he pushed that viewpoint to the limit. He even tried to turn the tables on the measurement problem by making the act of measurement central and to derive all of quantum physics by starting from the quantum jumps. things like states and vectors, (and things that operate on vectors to give states). Now, it is important to note that these are basically different things, namely

- vectors
- numbers
- operators

The same vector can also be written transposed, that is, written as a row of elements:

$$\mathbf{v}^T = [1, \quad 0, \quad 0] \quad (3.6)$$

Now a vector equals its transpose $\mathbf{v}^T = \mathbf{v}$ if it is not, technically, the same thing.

This would be trivial if QM was an ordinary mathematical theory. But it is not. Therefore I have to be careful from the very beginning. To correctly transform \mathbf{v} into \mathbf{v}^T I have to multiply it by a special vector, the unit vector \mathbf{I} .

I intuitively believe²⁵

References

Physics Today, April:40–46, 2009.

²⁵Intuitives usually do not *believe*, they think they *know*. I am taking the more humble scientific attitude, that I certainly do not know, I just, uhh, think I know.)

“That is, when we speak about velocity, we are not just talking about speed. Speed is a scalar (a number, in this instance, a real number). Velocity is speed in a certain direction. So the magnitude of the velocity vector might be 100 miles per hour and the direction of the velocity vector might be due south.” ?

Here is our “toolbox” of images that will help us understand (how and what we can understand) nature.

- Jeremy Bernstein. Max Born and the Quantum Theory. *Am. J. Phys.*, 73(11):999–1008, 2005.
- John Archibald Wheeler Charles W. Misner, Kip S. Thorne. *Gravitation*. W.H. Freeman and Company, San Francisco, 1970.
- P.A.M. Dirac. *The Principles of Quantum Mechanics*. Oxford University Press, London, 1958.
- Richard Feynman, Robert Leighton, and Matthew Sands. *The Feynman Lectures in Physics*. Addison-Wesley, Reading, Massachusetts, 1965a.
- Richard Feynman, Robert Leighton, and Matthew Sands. *The Feynman Lectures in Physics, Quantum Theory*, volume 3. Addison-Wesley, Reading, Massachusetts, 1965b.
- Martin Gutzwiller. Quantum Chaos. *Scholarpedia*, 2(12):3146, 2007.
- Wikipedia. Schroedinger wave equation. http://en.wikipedia.org/wiki/Introduction_to_quantum_mechanics#Schr.C3.B6dinger_wave_equation, August 2009.